Linear systems

In this section, X is a nonsingular projective variety over $k=\bar{k}$. In this setting, all Weil divisors are Cartier, so linear equiv. classes of Weil divisors are in correspondence with isom. classes of invertible sheaves. i.e.

$$\mathrm{Cl}\, X \cong \mathrm{Pic}\, X.$$

We will see that given any divisor D, we have another important correspondence:

$$\left\{\begin{matrix}\text{effective divisors } E \\ \text{s.t. } E \sim D\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\overset{\circ}{\cancel{s}} \in \Gamma(X, \mathcal{O}(D)) \\ \text{mod scaling}\end{matrix}\right\}$$

Def: Let $\mathcal{L}$ be an invertible sheaf on X and $s \in \Gamma(X,\mathcal{L})$. The divisor of zeros of s, $(s)_0$, is defined as follows: Let $\{U_i\}$ be an open cover of X s.t. $\mathcal{L}|_{U_i}$ is trivial, and fix isomorphisms

$$\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i}.$$

Set $s_i = \varphi_i(s) \in \Gamma(U_i, \mathcal{O}_{U_i})$, and $(s)_0 = \{(U_i, s_i)\}$.

We need to check that $(s)_0$ is actually a divisor. i.e. we need that s_i/s_j is regular (and a unit) on $U_i \cap U_j$. Consider the diagram below:

$$
\begin{tikzcd}
 & s\ \mathcal{L}(U_i) \arrow[dl, "\cong"'] \arrow[dr] & & \mathcal{L}(U_j)\ s \arrow[dl] \arrow[dr, "\cong"] & \\
s_i\ \mathcal{O}_X(U_i) \arrow[dr] & & s\ \mathcal{L}(U_i \cap U_j) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \mathcal{O}_X(U_j)\ s_j \arrow[dl] \\
 & \underset{s_i}{\mathcal{O}_X(U_i \cap U_j)} \arrow[rr, "\cong"] & & \underset{s_j}{\mathcal{O}_X(U_i \cap U_j)} &
\end{tikzcd}
$$

since this is an isomorphism, $s_j = u s_i$, u a unit. Thus, $s_i/s_j \in \mathcal{O}_X^*(U_i \cap U_j)$.

Thus, $(s)_0 = \{(U_i, s_i)\}$ defines a divisor, which is effective, since each s_i is regular on U_i.

Now we can prove the correspondence mentioned above:

Prop: Let D be a divisor and $\mathcal{L} = \mathcal{O}(D)$.

(a.) If $s \in \Gamma(X, \mathcal{L})$ nonzero, then $(s)_0 \sim D$.

(b.) If E is an effective divisor s.t. $E \sim D$, then $E = (s)_0$ for some $s \in \Gamma(X, \mathcal{L})$.

(c.) Let $s, s' \in \Gamma(X, \mathcal{L})$. Then $(s)_0 = (s')_0 \iff s' = \lambda s$ for some $\lambda \in k^*$.

Thus, we get the correspondence described above.

Pf: (a) D is a Cartier divisor, so we can write

$$D = \{(U_i, f_i)\}.$$

Then $\mathcal{O}(D)$ is a subsheaf of $\mathcal{K}$, so $s \in K^*$

On each U_i, $\mathcal{O}(D)|_{U_i} = \frac{1}{f_i}\mathcal{O}_{U_i}$. So we get isomorphisms

$$\varphi_i : \mathcal{O}(D)|_{U_i} \xrightarrow{\cdot f_i} \mathcal{O}_{U_i}$$

Thus, $(s)_0 = \{(U_i, sf_i)\} = D + (s)$, so $(s)_0 - D$ is principal and thus $(s)_0 \sim D$.

(b) $E \sim D \Rightarrow E = D + (f)$ some $f \in K^*$. In particular, $D + (f)$ is effective, so by definition, $f \in \Gamma(X, \mathcal{O}(D))$, and by the construction in part (a), $E = (f)_0$.

(c) If $(s)_0 = (s')_0$, then by part (a), we can think of $s, s' \in K^*$. Thus, using the above construction, we have local isomorphisms

$$\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\cdot f_i} \mathcal{O}_{U_i}$$

so on each U_i, $(sf_i) = (s'f_i) \Rightarrow (s/s') = 0$.

since (s/s') and $(s'/s) \geq 0$

Thus, $(s/s') = 0$ globally, so $s/s' \in \Gamma(X, \mathcal{O}_X^*) = k^*$. $\square$

since X is a proj. variety over k

Ex: Let $P = [0:1] \in \mathbb{P}^1$. Then $\mathcal{O}(P) \xrightarrow{\cdot x} \mathcal{O}(1)$. We can also describe P as the locally principal divisor

$$P = \{ (\underbrace{\operatorname{Spec} k[\tfrac{y}{x}]}_{=U_1}, 1), (\underbrace{\operatorname{Spec} k[\tfrac{x}{y}]}_{=U_2}, \tfrac{x}{y}) \}$$

Thus, $\mathcal{O}(P)|_{U_1} = \mathcal{O}_{U_1}$, and $\mathcal{O}(P)|_{U_2} = \frac{y}{x} \mathcal{O}_{U_2} \xrightarrow[\cdot \frac{x}{y}]{\cong} \mathcal{O}_{U_2}$

Take the global section $2x + y \in \Gamma(\mathbb{P}^1, \mathcal{O}(1))$.

This corresponds to $s = \frac{2x+y}{x} \in \Gamma(\mathbb{P}^1, \mathcal{O}(P)) \subseteq K^*$.

So $s_1 = 2 + y/x \in \Gamma(U_1, \mathcal{O}_{U_1})$, and

$s_2 = \frac{x}{y}\left(\frac{2x+y}{x}\right) = \frac{2x}{y} + 1 \in \Gamma(U_2, \mathcal{O}_{U_2})$

Globally, this corresponds to the point w/ homogeneous eqn $2x + y$, as desired, i.e. $[1:-2]$.

Def: A <u>complete linear system</u> on X is the set of all effective divisors linearly equivalent to some given divisor D, denoted $|D|$. By the Prop, $|D|$ is in one-to-one correspondence w/ $\mathbb{P}(\Gamma(X, \mathcal{O}(D)))$.

A <u>linear system</u> on X is the set of divisors corresponding to $\mathbb{P}(V)$, where $V \subseteq \Gamma(X, \mathcal{O}(D))$ is any subspace.

(All the dimensions are finite since $\Gamma(X, \mathcal{O}(D))$ is a f.d. k-vector space.)

Ex: If $L \subseteq \mathbb{P}^2_k$ is any line, then

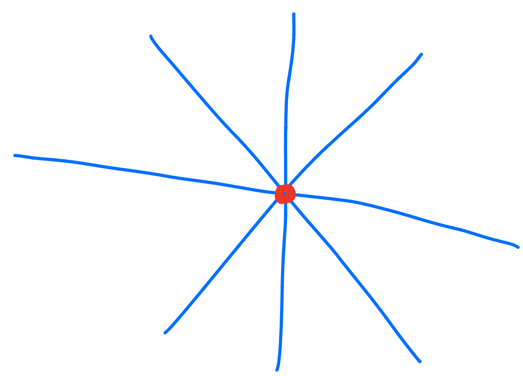
$$|L| = \left\{ \begin{array}{c} \text{effective divisors} \\ \text{of degree one} \end{array} \right\} = \{\text{lines in } \mathbb{P}^2\}$$

Of course these correspond to

$$\mathbb{P}\left(\Gamma(\mathbb{P}^2, \mathcal{O}(1))\right) = \{ax + by + cz\}/k \cong \mathbb{P}^2$$

(closed points)

If we take $V \subseteq \Gamma(\mathbb{P}^2, \mathcal{O}(1))$ to be lines through $[0:0:1]$, then $V = \{ax + by\}/k \cong \mathbb{P}^1$, an incomplete linear system.



Note that in this example, all the sections of V vanish at $[0:0:1]$. This is called a base point. More generally:

Def: $P \in X$ is a <u>base point</u> of a linear system δ if $P \in \operatorname{Supp} D$ for all $D \in \delta$. If δ has no base points, it is <u>base-point-free</u>. Equivalently, if $V \subseteq \Gamma(X, \mathcal{L})$ is the corresponding subspace, P is a base point iff $s_p \in m_p \mathcal{L}_p$ for all $s \in V$. In particular, δ is b.p.f iff $\mathcal{L}$ is generated by V (or a basis of V).

Thus, a morphism from $X \to \mathbb{P}^n$ is equivalent to a b.p.f.

linear system on X and a spanning set $s_0, \dots, s_n \in V$.

Remark: If δ is a b.p.f. linear system corr. to V, then the morphism corr. to δ is the one determined by a basis of V. Any other morphism will differ by an automorphism of $\mathbb{P}^n$.

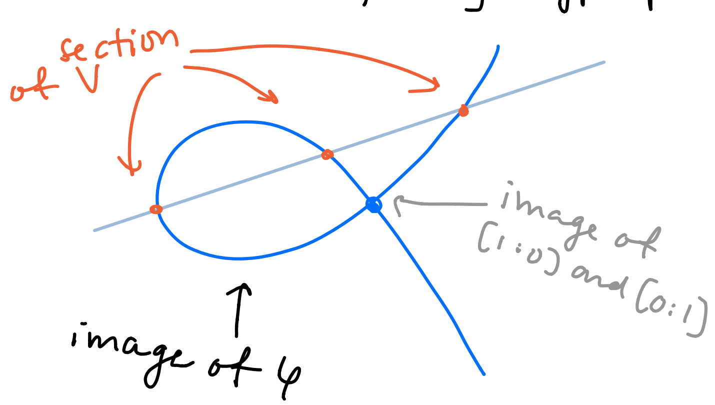
Geometric intuition: If $\varphi : X \to \mathbb{P}^n$ is determined by the b.p.f. linear system δ, the divisors in δ are the pull-backs of hyperplanes in $\mathbb{P}^n$

Ex: Let $V =$ span of s^3+t^3, s^2t, st^2 in $\Gamma(X, \mathcal{O}(3))$

$$\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$
$$[s:t] \longmapsto [s^3+t^3 : s^2t : st^2]$$

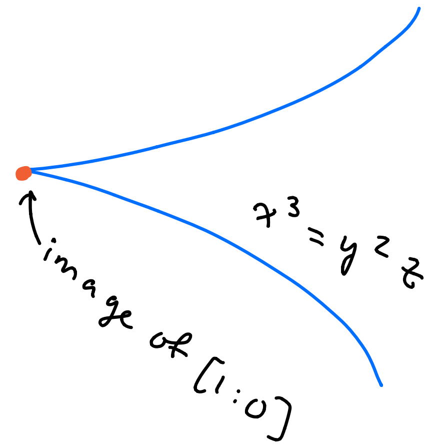
A hyperplane $ax+by+cz$ in $\mathbb{P}^2$ pulls back to the zero set of $a(s^3+t^3) + b(s^2t) + c(st^2)$, a section in V.

Notice that if $[0:1]$ is a zero of a section of V, then $a=0$, so $[1:0]$ is a section as well. That is, any hyperplane through $\varphi([0:1])$ also contains $\varphi([1:0])$. i.e. they have the same image $\Rightarrow \delta$ doesn't separate points



Ex: Let's look again at the cuspidal cubic:

$\varphi : [s:t] \longmapsto [st^2 : t^3 : s^3]$



Any section vanishing at $[1:0]$ is of the form
$a\, st^2 + b\, t^3 = t^2(as + bt)$
which has corresponding divisor

$2[1:0] + [-b:a]$. i.e. no section vanishes to order 1.

What is happening here?

Consider an affine open $z=1$, containing the cusp.
It intersects w/ the curve in $\operatorname{Spec} k[x,y]/(x^3-y^2)$.
Set $\mathfrak{m} = (x,y)$, corresponding to the cusp. Then

$$\mathfrak{m}/\mathfrak{m}^2 = (x,y)/(x,y)^2$$

is two dimensional. i.e. every line through the point is tangent to the curve at that point.

We can rephrase the conditions for when φ is a closed immersion in terms of linear systems:

Prop: If δ is b.p.f. and induces $\varphi : X \to \mathbb{P}^n$, then φ is a closed immersion $\iff$

1. If $P, Q \in X$, there is $D \in \delta$ s.t. $P \in \operatorname{Supp} D$, $Q \notin \operatorname{Supp} D$, and

closed pts

2. If $P \in X$, $t \in T_p(X) = (m_p/m_p^2)^\vee$, then there's $D \in \delta$ s.t. $P \in \operatorname{Supp} D$, but $t \notin T_p(D)$. i.e. there is some divisor D containing P s.t. t is not tangent to D at P.

Then in the above example, $T_p(X) = \left({}^{(t)}/_{(t)^2} \right)^\vee$, where $X = \operatorname{Spec} k[t]$, and for any divisor D containing P = the origin,

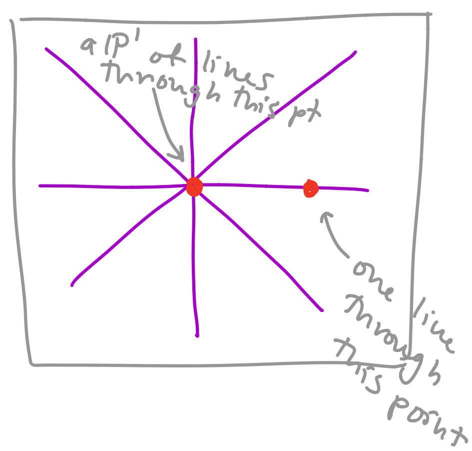
$$D = \operatorname{Spec} {}^{k[t]}/_{(t^2(a+bt))}$$

So $T_p(D) = \left({}^{(t)}/_{(t)^2} \right)^\vee = T_p(X)$, so it doesn't satisfy 2.

Ex: X = blowup of $\mathbb{P}^2$ at $[0:0:1]$. i.e.

$$X = \{ (p, \ell) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid p \in \ell \}$$

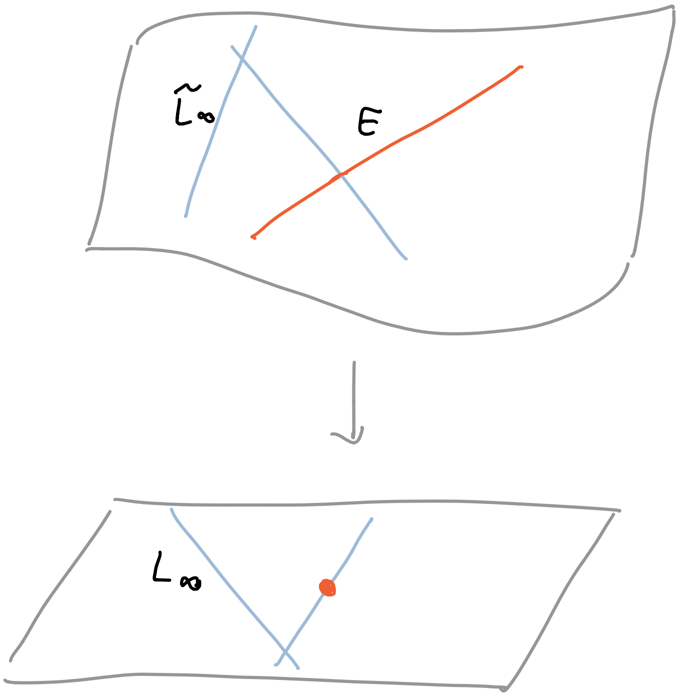
lines through $[0:0:1]$



If the $\mathbb{P}^2$ has homog. coordinates $[x:y:z]$ and the $\mathbb{P}^1$ has coords $[s:t]$, then X is defined by $xt - ys$ in $\mathbb{P}^2 \times \mathbb{P}^1$. X has three "obvious" maps to projective space:

1. Projection $\pi : X \to \mathbb{P}^2$, the "blowing down". π is an isomorphism away from the origin, and fiber over origin $= \overset{E}{\underset{\shortparallel}{\mathbb{P}^1}}$.

It is given by $\mathcal{L} = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, along w/ global sections x, y, z.

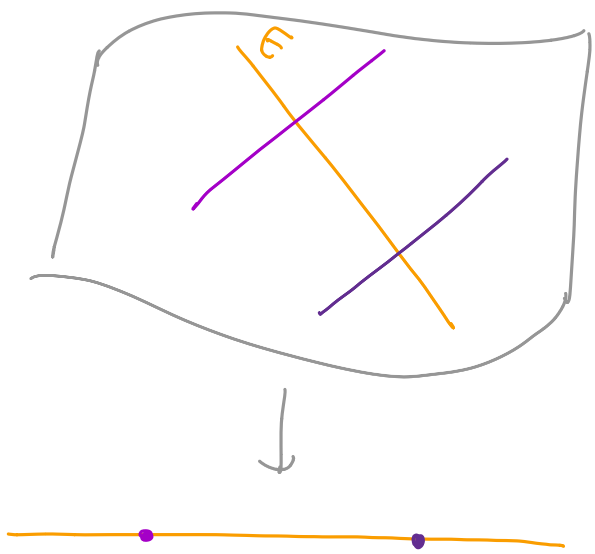


The linear system is the pull back of all lines. e.g. if $L_\infty = V(z)$ in $\mathbb{P}^2$, then it pulls back to $\tilde{L}_\infty$, an irreducible line in X. But $L_x = V(y)$ (the x-axis in the chart $\operatorname{Spec} k[x/z, y/z]$) pulls back to $E + \tilde{L}_x$, the sum of E, the <u>exceptional divisor</u>, and the closure of the generic point of L_x on X (check details here!). This linear system is denoted $|\pi^* H|$.

2. Projection $X \longrightarrow \mathbb{P}^1$, given by an invertible sheaf $\mathcal{M}$ and sections s, t. The linear system consists of lines corresponding to the fibers of the projection.

e.g. the section $bs - at$ of $\mathcal{O}_{\mathbb{P}^1}(1)$ (which defines the pt $[a:b]$) pulls back to the section $bs - at$ of $\mathcal{M}$ whose divisor of zeros is the line $([a:b],[a:b:z]) \subseteq X$.

The linear system is $|\tilde{L}_x| = |\pi^* H - E|$.



(3.) The composition of

$$X \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$$

Segre embedding corr. to inv. sheaf of type (1,1).

This is given by a very ample invertible sheaf $\mathcal{N}$ and global sections sx, sy, sz, tx, ty, tz (monomials of bi-degree (1,1)). The corresponding linear system is $|2\pi^* H - E|$. Note that $sy = tx$, so this maps into a hyperplane in $\mathbb{P}^5$. Think through the details of this!!!